Quasi-exact Solvability of

Dirac-Pauli Equation

and generalized Dirac oscillators

Choon-Lin Ho[1] and Pinaki Roy[2]

[1]*Department of Physics, Tamkang University, Tamsui 25137, Taiwan, R.O.C.*

[2]*Physics and Applied Mathematics Unit, Indian Statistical Institute, Calcutta 700035, India*

Abstract

In this paper we demonstrate that neutral Dirac particles in external electric fields, which are equivalent to generalized Dirac oscillators, are physical examples of quasi-exactly solvable systems. Electric field configurations permitting quasi-exact solvability of the system based on the $sl(2)$ symmetry are discussed separately in spherical, cylindrical, and Cartesian coordinates. Some exactly solvable field configurations are also exhibited.

PACS: 03.65.-w, 03.65.Pm, 02.30.Zz

Dec 5, 2003

I. Introduction

Search for exact solutions of wave equations, whether non relativistic or relativistic, has been there since the birth of quantum mechanics. In the case of non relativistic wave equations, i.e. Schrödinger equations, the class of potentials for which the complete spectrum or a part of the spectrum are known has grown over the years. However in the case of relativistic wave equations, in particular, the Dirac equation very few exactly solvable electromagnetic field configurations like homogeneous magnetic fields [1], homogeneous electrostatic fields [2], constant parallel magnetic fields [3] etc. are known.

It may be mentioned that while exact solvability is desirable, in practice it is not always possible to determine the whole spectrum. In non relativistic quantum mechanics a new class of potentials which are intermediate to exactly solvable ones and non solvable ones have been found recently. These are called quasi-exactly solvable (QES) problems for which it is possible to determine algebraically a part of the spectrum but not the whole spectrum [[4]-[9]]. Although there are exceptions, usually a QES problem admits a certain underlying Lie algebraic symmetry which is responsible for the quasi-exact solutions. For Dirac equations, it had been shown that Dirac equation with Coulomb interaction supplemented by a linear radial potential [10], and planar Dirac equation with Coulomb and homogeneous magnetic fields [11] are QES systems. More recently, the quasi-exact solvability of the Pauli equation was exhibited [12]. This immediately implies that the Dirac equation coupled minimally to a vector potential is QES, since the square of the Dirac Hamiltonian in this form is proportional to the Pauli equation, up to an additive constant, namely, the square of the rest energy.

Interactions of spin-1/2 fermions with electromagnetic fields are usually introduced in the Dirac equation following the minimal coupling prescription. However, this prescription is not the only possibility. For instance, the interaction due to anomalous magnetic moment is incorporated via the non-minimal coupling procedure, the resulting equation being called the Dirac-Pauli equation [13]. An interesting example is neutral fermions interacting with electromagnetic fields. There are a number of interesting phenomena involving non minimal coupling of neutral fermions to external electric fields. We recall that, the celebrated

Aharonov-Casher effect [14], which has been observed experimentally [15] is an example of such an interaction. Another interesting system involving neutral fermions is the Dirac oscillator, which is of considerable interest in quantum chromodynamics [16]. However, unlike the case with minimal coupling, studies in exact solutions of the Dirac equations with non-minimal couplings are rather scanty [17, 18, 19], not to mention studies in quasi-exact solutions of these systems. Only recently, efforts have been directed to exploring certain structure relating to such systems. For instance, the underlying supersymmetry (SUSY) of the Aharonov-Casher system [20], as well as that of a system of neutral fermions interacting with a central electric field were pointed out [21].

In the present paper we shall examine quasi-exact solvability of Dirac equation coupled non-minimally to external electric fields. In particular, based on the underlying $sl(2)$ symmetry we shall determine the forms of electric fields which give rise to QES Dirac-Pauli equations.

The organization of the paper is as follows. In Sect.II the Dirac-Pauli equation is presented, and its equivalence with the generalized Dirac oscillators demonstrated. Sect.III discusses the exact and quasi-exact solvability of the Dirac-Pauli equation with electric fields in the spherical coordinates. Sect.IV and V are devoted to the same problems with electric fields in the cylindrical and Cartesian coordinates, respectively. Sect.VI concludes the paper.

II. The Dirac-Pauli Equation for neutral Dirac particles

We consider the motion of a neutral fermion of spin-1/2 with mass m and an anomalous magnetic moment μ, in an external electromagnetic field described by the field strength $F_{\mu\nu}$. The fermion is described by a four-component spinor Ψ which obeys the Dirac–Pauli equation [13]

$$(i\gamma^\mu \partial_\mu - \frac{1}{2}\mu\sigma^{\mu\nu}F_{\mu\nu} - m)\Psi = 0 , \tag{1}$$

where $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$ are the Dirac matrices satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2}$$

3

with $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \ . \tag{3}$$

In terms of the external electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, one can rewrite the second term in eq.(1) as

$$\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} = i\boldsymbol{\alpha} \cdot \mathbf{E} - \boldsymbol{\Sigma} \cdot \mathbf{B} \ , \tag{4}$$

where $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, and $\Sigma^k = \frac{1}{2}\epsilon^{kij}\sigma^{ij}$. Here ϵ^{kij} is the totally anti-symmetric tensors with $\epsilon^{123} = 1$. For time-independent fields one may set

$$\Psi(t, \mathbf{r}) = e^{-i\mathcal{E}t}\psi(\mathbf{r}) \ , \tag{5}$$

and eq.(1) becomes

$$H\psi = \mathcal{E}\psi, \tag{6}$$

with the Hamiltonian H being given by

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + i\mu\boldsymbol{\gamma} \cdot \mathbf{E} - \mu\beta\boldsymbol{\Sigma} \cdot \mathbf{B} + \beta m \ , \tag{7}$$

where $\mathbf{p} = -i\nabla$ and $\beta = \gamma^0$.

In this paper we shall consider cases with only electric fields described by the the Hamiltonian

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + i\mu\boldsymbol{\gamma} \cdot \mathbf{E} + \beta m \ . \tag{8}$$

We choose the Dirac matrices in the standard representation

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{9}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices. We also define $\psi = (\chi, \varphi)^t$, where t denotes transpose, and both χ and φ are two-component spinors. Then the Dirac–Pauli equation becomes

$$\begin{aligned}
\boldsymbol{\sigma} \cdot (\mathbf{p} - i\mu\mathbf{E})\chi &= (\mathcal{E} + m)\varphi \ , \\
\boldsymbol{\sigma} \cdot (\mathbf{p} + i\mu\mathbf{E})\varphi &= (\mathcal{E} - m)\chi,
\end{aligned} \tag{10}$$

In the next three sections, we will demonstrate the exact and quasi-exact solvability of this set of equations in the spherical, cylindrical, and Cartesian coordinates.

Relation with the Dirac oscillator

Here we show that the above system is equivalent to the generalized Dirac oscillators. Since its introduction by Moshinsky and Szczepaniak in 1989, the original Dirac oscillator has attracted much attention in recent years [16]. It was introduced by replacing the momentum $\mathbf{p}$ in the field-free Dirac equation by $\mathbf{p} - im\omega\beta\mathbf{r}$, where m is the mass of the particle and ω is the oscillation frequency. The Hamiltonian of this system is thus given by

$$H = \boldsymbol{\alpha} \cdot (\mathbf{p} - im\omega\beta\mathbf{r}) + \beta m . \tag{11}$$

This system is exactly solvable.

Now if one compares eq.(11) with (8), one sees immediately that the Dirac oscillator is simply a special case of neutral Dirac particle in external electric field, if one makes the transformation

$$\mu\mathbf{E} \to m\omega\mathbf{r} . \tag{12}$$

That is, when the electric field $\mathbf{E}$ is proportional to the position vector $\mathbf{r}$. Hence, with a general $\mathbf{r}$-dependent electric field $\mathbf{E}(\mathbf{r})$, the system described by the Hamiltonian (8) can be viewed as generalizations of the original Dirac oscillator. In the discussions below, we shall concentrate on the neutral Dirac particles. But all conclusions thus obtained are applicable directly to the generalized Dirac oscillators.

III. Electric fields in spherical coordinates

First let us consider central electric field $\mathbf{E} = E_r\hat{\mathbf{r}}$. In this case, one can choose a complete set of observables to be $(H, \mathbf{J}^2, J_z, \mathbf{S}^2 = 3/4, K)$. Here $\mathbf{J}$ is the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$, where $\mathbf{L}$ is the orbital angular momentum, and $\mathbf{S} = \frac{1}{2}\boldsymbol{\Sigma}$ is the spin operator. The operator K is defined as $K = \gamma^0(\boldsymbol{\Sigma} \cdot \mathbf{L} + 1)$, which commutes with both H and $\mathbf{J}$.

Explicitly, we have

$$K = \text{diag}\left(\hat{k}, -\hat{k}\right) ,$$

$$\hat{k} = \boldsymbol{\sigma} \cdot \mathbf{L} + 1 . \tag{13}$$

The common eigenstates can be written as [22]

$$\psi = \frac{1}{r} \begin{pmatrix} f_-(r)\mathcal{Y}^k_{jm_j} \\ if_+(r)\mathcal{Y}^{-k}_{jm_j} \end{pmatrix} \tag{14}$$

here $\mathcal{Y}^k_{jm_j}(\theta,\phi)$ are the spin harmonics satisfying

$$\mathbf{J}^2\mathcal{Y}^k_{jm_j} = j(j+1)\mathcal{Y}^k_{jm_j} , \quad j = \frac{1}{2}, \frac{3}{2}, \cdots , \tag{15}$$

$$J_z\mathcal{Y}^k_{jm_j} = m_j\mathcal{Y}^k_{jm_j} , \quad |m_j| \leq j , \tag{16}$$

$$\hat{k}\mathcal{Y}^k_{jm_j} = -k\mathcal{Y}^k_{jm_j} , \quad k = \pm(j + \frac{1}{2}) , \tag{17}$$

and

$$(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})\mathcal{Y}^k_{jm_j} = -\mathcal{Y}^{-k}_{jm_j} , \tag{18}$$

where $\hat{\mathbf{r}}$ is the unit radial vector. Using the identity

$$\boldsymbol{\sigma} \cdot \mathbf{p} = i(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})\left(-\partial_r + \frac{1}{r}(\boldsymbol{\sigma} \cdot \mathbf{L})\right) , \tag{19}$$

one gets

$$\boldsymbol{\sigma} \cdot (\mathbf{p} \pm \mathbf{i}\mu\mathbf{E}) = i(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})\left(-\partial_r + \frac{K-1}{r} \pm \mu E_r\right) . \tag{20}$$

With these, eq.(10) reduces to

$$\left(\frac{d}{dr} + \frac{k}{r} + \mu E_r\right) f_- = (\mathcal{E} + m) f_+ , \tag{21}$$

$$\left(-\frac{d}{dr} + \frac{k}{r} + \mu E_r\right) f_+ = (\mathcal{E} - m) f_- . \tag{22}$$

This shows that f_- and f_+ forms a one-dimensional SUSY pairs [21]. For if we write the superpotential W as

$$W = \frac{k}{r} + \mu E_r , \tag{23}$$

6

then eqs.(21) and (22) become

$$A^- A^+ f_- = \left(\mathcal{E}^2 - m^2 \right) f_- , \tag{24}$$

$$A^+ A^- f_+ = \left(\mathcal{E}^2 - m^2 \right) f_+ , \tag{25}$$

with

$$A^\pm \equiv \pm \frac{d}{dr} + W . \tag{26}$$

Explicitly, the above equations read

$$\left(-\frac{d^2}{dr^2} + W^2 \mp W' \right) f_\mp = \left(\mathcal{E}^2 - m^2 \right) f_\mp . \tag{27}$$

Here and below the prime means differentiation with respect to the basic variable. Eq.(27) clearly exhibits the SUSY structure of the system. The operators acting on $f_\pm$ in eq.(27) are said to be factorizable, i.e. as products of A^- and A^+. The ground state, with $\mathcal{E}^2 = m^2$, is given by one of the following two sets of equations:

$$A^+ f_-^{(0)}(r) = 0 , \quad f_+^{(0)}(r) = 0 ; \tag{28}$$

$$A^- f_+^{(0)}(r) = 0 , \quad f_-^{(0)}(r) = 0 , \tag{29}$$

depending on which solution is normalizable. The solutions are generally given by

$$f_\mp \propto r^{\mp k} \exp \left(\mp \int dr \mu E_r \right) . \tag{30}$$

We now classify the forms of the electric field $E_r(r)$ which allow exact and quasi-exact solutions. To be specific, we consider the situation where $k < 0$ and $\int dr \mu E_r > 0$, so that $f_-^{(0)}$ is normalizable, and $f_+^{(0)} = 0$. The other situation can be discussed similarly. In this case, eq.(23) becomes

$$W = -\frac{|k|}{r} + \mu E_r . \tag{31}$$

We determine the forms of E_r that give exact/quasi-exact energy $\mathcal{E}$ and the corresponding function f_-. The corresponding function f_+ is obtained using eq.(21).

7

A. Exactly solvable cases

Comparing the forms of the superpotential W in eq.(31) with Table (4.1) in [23], one concludes that there are three forms of E_r giving exact solutions of the problem :

      i) oscillator-like : $\mu E_r(r) \propto r$;

      ii) Coulomb potential-like : $\mu E_r(r) \propto$ constant ;

      iii) zero field-like : $\mu E_r(r) \propto 1/r$.

Case (i) and (ii) had been considered in [18] and [19], and case (iii) in [18].

We mention here that the case with oscillator-like field, i.e. case (i), is none other than the spherical Dirac oscillator.

B. Quasi-exactly solvable cases

The form of the superpotential W in eq.(31) fits into three classes, namely, Classes VII, VIII and IX, of $sl(2)$-based QES systems in [5]. Electric field configurations permitting any number of solvable excited states in each of these classes can be constructed according to the general procedure given in [12]. The procedure makes use of the connection between quasi-exact solvability and the SUSY structure (or, equivalently, the factorizability) of the systems. Below we describe the construction of QES E_r for the simplest cases in Class VII. Other classes can be considered similarly.

An outline of the method of construction

Here we describe briefly the main ideas underlying the construction of QES systems as given in [12]. As before, we shall concentrate only on solution of the upper component f_-, which is assumed to have a normalizable zero energy state.

Eq.(27) shows that f_- satisfies the Schrödinger equation $H_- f_- = \epsilon f_-$, with energy parameter $\epsilon \equiv \mathcal{E}^2 - m^2$, and

$$
\begin{aligned}
H_- &= A^- A^+ \\
&= -\frac{d^2}{dr^2} + V(r) ,
\end{aligned}
\tag{32}
$$

with

$$V(r) = W(r)^2 - W'(r) \ . \tag{33}$$

We shall look for $V(r)$ such that the system is QES. According to the theory of QES models, one first makes an imaginary gauge transformation on the function f_-

$$f_-(r) = \phi(r)e^{-g(r)} \ , \tag{34}$$

where $g(r)$ is called the gauge function. The function $\phi(r)$ satisfies

$$-\frac{d^2\phi(r)}{dr^2} + 2g'\frac{d\phi(r)}{dr} + \left[V(r) + g'' - g'^2\right]\phi(r) = \epsilon\phi(r) \ . \tag{35}$$

For physical systems which we are interested in, the phase factor $\exp(-g(r))$ is responsible for the asymptotic behaviors of the wave function so as to ensure normalizability. The function $\phi(r)$ satisfies a Schrödinger equation with a gauge transformed Hamiltonian

$$H_G = -\frac{d^2}{dr^2} + 2W_0(r)\frac{d}{dr} + \left[V(r) + W_0' - W_0^2\right] \ , \tag{36}$$

where $W_0(r) = g'(r)$. Now if $V(r)$ is such that the quantal system is QES, that means the gauge transformed Hamiltonian H_G can be written as a quadratic combination of the generators J^a of some Lie algebra with a finite dimensional representation. Within this finite dimensional Hilbert space the Hamiltonian H_G can be diagonalized, and therefore a finite number of eigenstates are solvable. For one-dimensional QES systems the most general Lie algebra is $sl(2)$ ([4]-[9]). Hence if eq.(36) is QES then it can be expressed as

$$H_G = \sum C_{ab}J^aJ^b + \sum C_aJ^a + \text{constant} \ , \tag{37}$$

where C_{ab}, C_a are constant coefficients, and the J^a are the generators of the Lie algebra $sl(2)$ given by

$$\begin{aligned} J^+ &= z^2\frac{d}{dz} - Nz \ , \\ J^0 &= z\frac{d}{dz} - \frac{N}{2} \ , && N = 0, 1, 2\ldots \\ J^- &= \frac{d}{dz} \ . \end{aligned} \tag{38}$$

9

Here the variables r and z are related by $z = h(r)$, where $h(\cdot)$ is some (explicit or implicit) function . The value $j = N/2$ is called the weight of the differential representation of $sl(2)$ algebra, and N is the degree of the eigenfunctions ϕ, which are polynomials in a $(N+1)$-dimensional Hilbert space

$$\phi = (z - z_1)(z - z_2) \cdots (z - z_N) . \tag{39}$$

The requirement in eq.(37) fixes $V(r)$ and $W_0(r)$, and H_G will have an algebraic sector with $N+1$ eigenvalues and eigenfunctions. For definiteness, we shall denote the potential V admitting $N+1$ QES states by V_N. From eqs.(34) and (39), the function f_- in this sector has the general form

$$f_- = (z - z_1)(z - z_2) \cdots (z - z_N) \exp\left(- \int^z W_0(r) dr\right) , \tag{40}$$

where z_i $(i = 1, 2, \ldots, N)$ are N parameters that can be determined by plugging eq.(40) into eq.(35). The algebraic equations so obtained are called the Bethe ansatz equations corresponding to the QES problem [9, 11]. Now one can rewrite eq.(40) as

$$f_- = \exp\left(- \int^z W_N(r, \{z_i\}) dr\right) , \tag{41}$$

and

$$W_N(r, \{z_i\}) = W_0(r) - \sum_{i=1}^{N} \frac{h'(r)}{h(r) - z_i} . \tag{42}$$

There are $N+1$ possible functions $W_N(r, \{z_i\})$ for the $N+1$ sets of eigenfunctions ϕ. Inserting eq.(41) into $H_- f_- = \epsilon f_-$, one sees that W_N satisfies the Ricatti equation [8, 24]

$$W_N^2 - W_N' = V_N - \epsilon_N , \tag{43}$$

where ϵ_N is the energy parameter corresponding to the eigenfunction f_- given in eq.(40) for a particular set of N parameters $\{z_i\}$.

From eqs.(32), (33) and (43) it is clear how one should proceed to determine the electric fields so that the Dirac-Pauli equation becomes QES based on $sl(2)$: one needs only to determine the superpotentials $W(r)$ according to eq.(43) from the QES potentials $V(r)$

10

classified in [5]. This is easily done by observing that the superpotential W_0 corresponding to $N = 0$ is related to the gauge function $g(r)$ associated with a particular class of QES potential $V(r)$ by $g'(r) = W_0(r)$. Once W_0 is obtained, the corresponding electric field $E^{(0)}$ is obtained through eq.(31):

$$\mu E_r^{(0)} = W_0 + |k|/r \ . \tag{44}$$

This is the required electric field that allows the weight zero ($j = N = 0$) state, i.e. the ground state, to be known in that class. The more interesting task is to obtain higher weight states (i.e. $j > 0$), which will include excited states. For weight j ($N = 2j$) states, this is achieved by forming the superpotential $W_N(r, \{z_i\})$ according to eq.(42). Of the $N + 1$ possible sets of solutions of the Bethe ansatz equations, the set of roots $\{z_1, z_2, \ldots, z_N\}$ to be used in eq.(42) is chosen to be the set for which the energy parameter of the corresponding state is the lowest (usually it is the ground state). The required electric field which gives rise to the $N + 1$ solvable states is then obtained as

$$\mu E_r^{(N)} = W_N + |k|/r \ . \tag{45}$$

For the spherically symmetric electric fields which we consider in this section, there are three possible types of $sl(2)$-based QES field configurations. They belong to Class VII, VIII, and IX in the classification listed in [5]. Below we shall illustrate our construction of QES electric fields through Class VII QES systems.

Class VII

The general potential in Class VII has the form

$$V_N(r) = a^2 r^6 + 2ab r^4 + \left[b^2 - a\left(4N + 2\gamma + 3\right)\right] r^2 + \gamma\left(\gamma - 1\right) r^{-2} - b\left(2\gamma + 1\right) \ , \tag{46}$$

where a, b and γ are constants. The gauge function is

$$g(r) = \frac{a}{4}r^4 + \frac{b}{2}r^2 - \gamma \ln r \ . \tag{47}$$

11

We must have $a, \gamma > 0$ to ensure normalizability of the wave function. Eqs.(47) and (31), together with the relation $W_0(r) = g'(r)$, give us the electric field $E_r^{(0)}$:

$$\mu E_r^{(0)}(r) = ar^3 + br \ . \tag{48}$$

The Dirac-Pauli equation with this field configuration admits a QES ground state with energy $\mathcal{E}^2 = m^2$ ($\epsilon = 0$) and ground state function $f_- \propto \exp(-g_0(r))$. Also, here we have $\gamma = |k|$. We retain the symbol γ so that some general formulae in this section can be carried over in the next section simply with γ redefined.

To determine electric field configurations admitting QES potentials V_N with higher weight, we need to obtain the Bethe ansatz equations for ϕ. Letting $z = h(r) = r^2$, eq.(35) becomes

$$\left[-4z \frac{d^2}{dz^2} + \left(4az^2 + 4bz - 2\left(2\gamma + 1 \right) \right) \frac{d}{dz} - \left(4aNz + \epsilon \right) \right] \phi(z) = 0 \ . \tag{49}$$

In terms of the $sl(2)$ generators J^+, J^- and J^0, the differential operator acting on $\phi(z)$ in eq.(49) can be written as

$$T_{VII} = -4J^0 J^- + 4aJ^+ + 4bJ^0 - 2\left(N + 2\gamma + 1 \right) J^- + \text{constant} \ . \tag{50}$$

For $N = 0$, the value of the ϵ is $\epsilon = 0$. For higher $N > 0$ and $\phi(r) = \prod_{i=1}^N (z - z_i)$, the electric field $E_r^{(N)}(r)$ is obtained from eqs.(42) and (45):

$$\mu E_r^{(N)}(r) = \mu E_r^{(0)}(r) - \sum_{i=1}^N \frac{h'(r)}{h(r) - z_i} \ . \tag{51}$$

For the present case, the roots z_i's are found from the Bethe ansatz equations

$$2az_i^2 + 2bz_i - \left(2\gamma + 1 \right) - \sum_{l \neq i} \frac{z_i}{z_i - z_l} = 0 \ , \qquad i = 1, \ldots, N \ , \tag{52}$$

and ϵ in terms of the roots z_i's is

$$\epsilon = 2\left(2\gamma + 1 \right) \sum_{i=1}^N \frac{1}{z_i} \ . \tag{53}$$

For $N = 1$ the roots z_1 are

$$z_1^{\pm} = \frac{-b \pm \sqrt{b^2 + 2a(2\gamma + 1)}}{2a} \ , \tag{54}$$

12

and the values of ϵ are

$$\epsilon^{\pm} = 2\left(b \pm \sqrt{b^2 + 2a(2\gamma + 1)}\right) . \tag{55}$$

For $a > 0$, the root $z_1^- = -|z_1^-| < 0$ gives the ground state. With this root, one gets the superpotential

$$W_1(r) = ar^3 + br - \frac{2r}{r^2 + |z_1^-|} - \frac{\gamma}{r} . \tag{56}$$

From eq.(51), the corresponding electric field is

$$\mu E_r^{(1)}(r) = ar^3 + br - \frac{2r}{r^2 + |z_1^-|} . \tag{57}$$

The QES potential appropriate for the problem is

$$
\begin{aligned}
V(x) &= W_1^2 - W_1' , \\
&= V_1 - \epsilon . \tag{58}
\end{aligned}
$$

The one-dimensional SUSY always sets the energy parameter of ground state at $\epsilon = 0$. Hence, the ground state and the excited state have energy parameter $\epsilon = 0$ and $\epsilon = \epsilon^+ - \epsilon^- = 4\sqrt{b^2 + 2a(2\gamma + 1)}$, and wave function

$$f_- \propto e^{-g_0(r)}\left(r^2 - z_1^-\right) \tag{59}$$

and

$$f_- \propto e^{-g_0(r)}\left(r^2 - z_1^+\right) , \tag{60}$$

respectively.

QES potentials and electric fields for higher degree N can be constructed in the same manner.

IV. Electric fields in cylindrical coordinates

We now treat the Dirac-Pauli equation in the cylindrical coordinates. In this case, it is easier to solve $H^2\Phi = \mathcal{E}^2\Phi$ instead of $H\psi = \mathcal{E}\psi$. Once Φ is solved, the desired solution ψ is

13

given by $\psi = (H + \mathcal{E})\Phi$. Setting $\Phi = (f_-, f_+)^t$, the equation $(H^2 - \mathcal{E}^2)\Phi = 0$ becomes

$$\left[p^2 \mp \mu \nabla \cdot \mathbf{E} + \mu^2 E^2 \pm i\mu\boldsymbol{\sigma} \cdot (2\mathbf{E} \times \nabla - \nabla \times \mathbf{E}) \right] f_\mp = \left(\mathcal{E}^2 - m^2 \right) f_\mp . \tag{61}$$

We consider electric field configuration which has cylindrical symmetry, namely,

$$E_x = xf(\rho) , \quad E_y = yf(\rho) , \quad E_z = 0 , \tag{62}$$

where $\rho = \sqrt{x^2 + y^2}$, and $f(\rho)$ is some function of ρ. It can be checked that the conserved quantities are p_z and $J_z = L_z + \Sigma_3/2$. Thus the eigenstates can be chosen as

$$f_- = \frac{e^{ik_z z}}{\sqrt{\rho}} \begin{pmatrix} R_1(\rho)e^{il\phi} \\ R_2(\rho)e^{i(l+1)\phi} \end{pmatrix} , \quad f_+ = \frac{e^{ik_z z}}{\sqrt{\rho}} \begin{pmatrix} R_3(\rho)e^{il\phi} \\ R_4(\rho)e^{i(l+1)\phi} \end{pmatrix} . \tag{63}$$

Φ is an eigenstate of p_z and J_z with eigenvalues k_z (k_z real) and $j_z = l+1/2$ ($l = 0, \pm 1, \pm 2 \ldots$), respectively. Eq.(61) reduce to four decoupled ones:

$$\left[-\frac{d^2}{d\rho^2} + \mu\rho^2 f^2 - 2\mu f (l+1) - \mu\rho f' + \frac{l^2 - \frac{1}{4}}{\rho^2} \right] R_1(\rho) = \epsilon R_1(\rho) , \tag{64}$$

$$\left[-\frac{d^2}{d\rho^2} + \mu\rho^2 f^2 + 2\mu f l - \mu\rho f' + \frac{(l+1)^2 - \frac{1}{4}}{\rho^2} \right] R_2(\rho) = \epsilon R_2(\rho) , \tag{65}$$

$$\left[-\frac{d^2}{d\rho^2} + \mu\rho^2 f^2 + 2\mu f (l+1) + \mu\rho f' + \frac{l^2 - \frac{1}{4}}{\rho^2} \right] R_3(\rho) = \epsilon R_3(\rho) , \tag{66}$$

$$\left[-\frac{d^2}{d\rho^2} + \mu\rho^2 f^2 - 2\mu f l + \mu\rho f' + \frac{(l+1)^2 - \frac{1}{4}}{\rho^2} \right] R_4(\rho) = \epsilon R_4(\rho) , \tag{67}$$

where $\epsilon \equiv \mathcal{E}^2 - m^2 - k_z^2$. From the above equations it follows that eqs.(64) and (67) represent a pair of one-dimensional SUSY partners [12], with superpotential given by

$$W(\rho) = \mu\rho f(\rho) - \frac{\gamma}{\rho} , \quad \gamma = |l| + 1/2 . \tag{68}$$

Eqs.(65) and (66) form another pair of SUSY partners, obtainable from eqs.(67) and (64), respectively, by changing μ to $-\mu$. Results in [12] can therefore be carried over directly with minor modifications.

To be specific, let us assume $\mu f > 0$. For $l \geq 0$, we have $R_2 = R_3 = 0$. The ground state is given by

$$R_1 \propto \exp(-g(\rho)) , \tag{69}$$

$$g(\rho) = \mu \int^\rho \rho' f(\rho')d\rho' - \gamma \ln \rho , \tag{70}$$

14

and $R_4 = 0$. For excited states, the components R_1 and R_4 are related by $R_4 \propto A^+ R_1$, where A^+ is defined by eq.(26) and (68). For $l \leq -1$, we simply interchange R_1 and R_2, and R_3 and R_4.

Comparing eq.(68) with (31), it is seen that all results in Sect. III can be carried over by changing r, E_r and $\gamma = |k|$ in eq.(31) to ρ, ρf, and $\gamma = |l| + 1/2$, respectively. Hence we will not discuss this case in details, but only quote some main results. For instance, the exactly solvable cases are:

    i) oscillator-like : $\mu f(\rho) \propto$ constant ;

    ii) Coulomb potential-like : $\mu f(\rho) \propto 1/\rho$;

    iii) zero field-like : $\mu f(\rho) \propto 1/\rho^2$.

As in Sect III, the quasi-exactly solvable cases are Classes VII, VIII and IX of QES systems based on the $sl(2)$ algebra as classified by Turbiner [5]. For the case of Class VII, the simplest two configurations of the electric fields are obtained from the function $f(\rho)$ as follows:

    i) one-state case: $\mu f^{(0)}(\rho) = a\rho^2 + b \ (a > 0)$;

    ii) two-state case: $\mu f^{(1)}(\rho) = a\rho^2 + b - 2/(\rho^2 + |z_1^-|)$, where z_1^- is given by eq.(54).

V. Electric fields in Cartesian coordinates

Finally we consider electric field configurations in one dimension. For definiteness, we assume

$$E_x(x) = \bar{W}(x) , \quad E_y = E_z = 0 . \tag{71}$$

The conserved quantities in this case are p_y and p_z, with eigenvalues k_y and k_z, respectively. Eigenstate can be written as

$$\psi = e^{ik_y y + ik_z z} \begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix} . \tag{72}$$

Substituting eqs.(71) and (72) into (61), we get

$$\left[-\frac{d^2}{dx^2} + \left(\mu \bar{W}(x) \mp \sigma_3 k_y \right)^2 \mp \mu \bar{W}' \right] f_\mp = \left(\mathcal{E}^2 - m^2 - k_z^2 \right) f_\mp . \tag{73}$$

Again, these equations are in the same form as eq.(3) in [12] and eq.(408) in [23]. Similar to the case in Section IV, the four components of ψ are related by two one-dimensional SUSY, namely, the 1st (2nd) component of f_- and the 2nd (1st) component of f_+ form a one-dimensional SUSY pair. As such, this system can be dealt with according to the general procedures outlined in the last section and in [12]. Here we shall only give the classifications of the electric field configurations which permit exact and quasi-exact solutions.

A. Exactly solvable cases

The exactly solvable cases have been classified in [23]. There are three types of exactly solvable field configurations, namely,

$$\mu\bar{W} \mp k_y = \begin{cases} \alpha x + c & \text{(shifted oscillator-like) ;} \\ c_1 \exp(-\alpha x) + c_2 & \text{(Morse-potential) ;} \\ \tanh(\alpha x) + c & \text{(Rosen-Morse II) ;} \end{cases} \tag{74}$$

Here c, c_1, c_2 and α are constants.

B. Quasi-exactly solvable cases

From the discussions given in [12], the system of eq.(73) are QES based on the $sl(2)$ algebra. In fact, six classes of QES electric field configurations can be constructed. These classes correspond to Class I to Class VI in Turbiner's classification. One can follow the general procedures described in Sect. III to construct in each class the electric fields which admit different numbers of QES states, including excited states. Here we only present the main results for Class I, referring the readers to [12] for further details.

Class I

We concentrate on the first set of one-dimensional SUSY system in eq.(73), namely, the first component of f_- and the second component of f_+. The superpotential for the system is

$$W(x) \equiv \mu\bar{W}(x) - k_y . \tag{75}$$

16

According to Turbiner's classification, the QES potential belonging to Class I has the form

$$V_N(x) = a^2 e^{-2\alpha x} - a\left[\alpha(2N+1) + 2b\right] e^{-\alpha x} + c\left(2b - \alpha\right) e^{\alpha x} + c^2 e^{2\alpha x} + b^2 - 2ac . \qquad (76)$$

Without loss of generality, we assume $\alpha, a, c > 0$ for definiteness. The corresponding gauge function $g(x)$ is given by

$$g(x) = \frac{a}{\alpha} e^{-\alpha x} + \frac{c}{\alpha} e^{\alpha x} + bx . \qquad (77)$$

One should always keep in mind that the parameters selected must ensure convergence of the function $\exp(-g(x))$ in order to guarantee normalizability of the wave function . The potential $V(x)$ that gives the ground state, with energy parameter $\epsilon \equiv \mathcal{E}^2 - m^2 - k_z^2 = 0$, is generated by

$$
\begin{aligned}
V(x) &= V_0 \\
&= W_0^2 - W_0' ,
\end{aligned}
\qquad (78)
$$

with

$$
\begin{aligned}
W_0(x) &= g'(x) \\
&= -ae^{-\alpha x} + ce^{\alpha x} + b .
\end{aligned}
\qquad (79)
$$

The corresponding electric field is

$$\mu E_x^{(0)} = -a\alpha e^{-\alpha x} + c\alpha e^{\alpha x} + b + k_y . \qquad (80)$$

To obtain electric fields and the corresponding potentials which admit solvable states with higher weights j, we must first derive the Bethe ansatz equations. To this end, let us perform the change of variable $z = h(x) = \exp(-\alpha x)$. Eq.(35) then becomes

$$\left\{ -\alpha z^2 \frac{d^2}{dz^2} + \left[2az^2 - (2b+\alpha)z - 2c\right]\frac{d}{dz} + \left[-2aNz - \frac{\epsilon}{\alpha}\right] \right\} \phi(z) = 0 . \qquad (81)$$

The differential operator acting on $\phi(z)$ can be written as a quadratic combination of the $sl(2)$ generators J^+, J^- and J^0 as

$$T_I = -\alpha J^+ J^- + 2aJ^+ - \left[\alpha(N+1) + 2b\right] J^0 - 2cJ^- + \text{constant} . \qquad (82)$$

17

For $N > 0$, there are $N + 1$ solutions which include excited states. Assuming $\phi(z) = \prod_{i=1}^{N}(z - z_i)$ in eq.(81), one obtains the Bethe ansatz equations which determine the roots z_i's

$$2az_i^2 - (2b + \alpha)z_i - 2c - 2\alpha \sum_{l \neq i} \frac{z_i^2}{z_i - z_l} = 0 , \qquad i = 1, \ldots, N , \tag{83}$$

and the equation which gives the energy parameter in terms of the roots z_i's

$$\epsilon = 2\alpha c \sum_{i=1}^{N} \frac{1}{z_i} . \tag{84}$$

Each set of $\{z_i\}$ determine a QES energy E with the corresponding polynomial ϕ.

As an example, consider the $j = 1/2$ case with $N = 1$ and $\phi(z) = z - z_1$. There are two solutions. From eq.(83), one sees that the root z_1 satisfies

$$2az_1^2 - (2b + \alpha)z_1 - 2c = 0 , \tag{85}$$

which gives two solutions

$$z_1^{\pm} = \frac{(2b + \alpha) \pm \sqrt{(2b + \alpha)^2 + 16ac}}{4a} . \tag{86}$$

The corresponding energy parameters are

$$\begin{aligned} \epsilon^{\pm} &= 2\alpha c \frac{1}{z_1} \\ &= -\frac{\alpha}{2}\left[(2b + \alpha) \mp \sqrt{(2b + \alpha)^2 + 16ac}\right] . \end{aligned} \tag{87}$$

For the parameters assumed here, the solution with root $z_1^- = -|z_1^-| < 0$ gives the ground state, while that with root $z_1^+ > 0$ gives the first excited state. The superpotential W_1 is constructed according to eq.(42)

$$\begin{aligned} W_1(x) &= W_0 - \frac{h'(x)}{h(x) - z_1^-} \\ &= -ae^{-\alpha x} + ce^{\alpha x} + \frac{\alpha}{1 + |z_1^-|e^{\alpha x}} + b . \end{aligned} \tag{88}$$

This gives the electric field

$$\mu E_x^{(1)}(x) = -ae^{-\alpha x} + ce^{\alpha x} + \frac{\alpha}{1 + |z_1^-|e^{\alpha x}} + b + k_y . \tag{89}$$

18

The ground state and the excited state have energy $\mathcal{E}^2 = m^2 + k_z^2$ and $\mathcal{E}^2 = m^2 + k_z^2 + \alpha\sqrt{(2b+\alpha)^2 + 16ac}$, respectively.

VI. Conclusions

In this paper we have shown that the Dirac-Pauli equation describing a neutral particle with anomalous magnetic moment in an external electric field is a QES system. Forms of electric field configurations permitting exact solutions, and QES states based on $sl(2)$ algebra are classified in the spherical, cylindrical, and Cartesian coordinates.

We have also demonstrated that the Dirac-Pauli equation with only electric fields are equivalent to generalized Dirac oscillators. Thus all the results obtained here are directly applicable to the later. Particularly, the exact solvability of the original Dirac oscillator is now recognized as one of the three exactly solvable cases of the Dirac-Pauli equation.

It has been shown that the two-dimensional Dirac-Pauli equation is equivalent to a two-dimensional Dirac equation minimally coupled to a vector potential by a duality transformation [14, 25]. However, as mentioned in Sect.I, the square of the Dirac Hamiltonian minimally coupled to a vector potential is proportional to the Pauli equation up to an additive constant. Hence the exact solvability discussed in [23] and the quasi-exactly solvability discussed in [12] of the Pauli equation can be straightforwardly applied to the two-dimensional Dirac-Pauli equation.

References

[1] I. Rabi, Z. Phys. **49**, 507 (1928).

[2] F. Sauter, Z. Phys. **69**, 742 (1931).

[3] L. Lam, Phys. Lett. **A31**, 406 (1970).

[4] A. Turbiner and A.G. Ushveridze, Phys. Lett. **A126**, 181 (1987).

[5] A. Turbiner, Comm. Math. Phys. **118**, 467 (1988).

[6] N. Kamran and P. Olver, J. Math. Anal. Appl. **145**, 342 (1990); A. González, N. Kamran and P.J. Olver, Comm. Math. Phys. **153**, 117 (1993); Contemp. Math. Phys. **160**, 113 (1994).

[7] M.A. Shifman and A.V. Turbiner, Comm. Math. Phys. **126**, 347 (1989); M.A. Shifman, Contemp. Math. **160**, 237 (1994); A.V.Turbiner, Contemp. Math. **160**, 263 (1994); G. Post and A.V. Turbiner, Russ. J. Math. Phys. **3**, 113 (1995).

[8] M.A. Shifman, Int. J. Mod. Phys. **A4**, 2897; 3305 (1989).

[9] A.G. Ushveridze, *Sov. Phys.-Lebedev Inst. Rep.* **2**, 50, 54 (1988); *Quasi-exactly solvable models in quantum mechanics* (IOP, Bristol, 1994).

[10] Y. Brihaye and P. Kosinski, Mod. Phys. Lett. **A13**, 1445 (1998).

[11] C.-L. Ho and V.R. Khalilov, Phys. Rev. **A61**, 032104 (2000); C.-M. Chiang and C.-L. Ho, Phys. Rev. **A63** 062105 (2001); J. Math. Phys. **43**, 43 (2002).

[12] C.L. Ho and P. Roy, J. Phys. **A36**, 4617 (2003).

[13] W. Pauli, Rev. Mod. Phys. **13**, 203 (1941).

[14] Y. Aharonov and A. Casher, Phys. Rev. Lett. **53**, 319 (1984). A. S. Goldhaber, Phys. Rev. Lett. **62**, 482 (1989); T. H. Boyer, Phys. Rev. A **36**, 5083 (1987); Y. Aharonov, P. Pearle, and L. Vaidman, Phys. Rev. A **37**, 4052 (1988).

[15] C. Cimmino, G. I. Opat, A. G. Klein, H. Kaiser, S. A. Werner, M. Arif, and R. Clothier, Phys. Rev. Lett. **63**, 380 (1989).

[16] M. Moshinsky and A. Szczepanaik, J. Phys. bf A22, L817 (1989); M. Moreno and A. Zentalla, *ibid.* L821 (1989); J. Benitez, R.P Martinez y Romero, H.N. Nunez-Yepez and A.L. Salas-Brito, Phys. Rev. Lett. **64**, 1643 (1990);C. Quesne and M. Moshinsky, J. Phys. **A23**, 2263 (1990); O.L. de Lange, J. Phys. **A24**, 667 (1991).

[17] R.F. O'Connell, Phys. Rev. **176**, 1433 (1968).

[18] G.V. Shishkin and V.M. Villalba, J. Math. Phys. **34**, 5037 (1993).

[19] Q.-L. Lin, Phys. Rev. **A61**, 022101 (2000).

[20] S. Bruce, L. Roa, C. Saavedra and A.B. Klimov, preprint quant-ph/9905074 (1999).

[21] V.V. Semenov, J.Phys. **A23**, L721 (1990).

[22] F. Gross, *Relativistic Quantum Mechanics and Field Theory* (John Wiley & Sons, New York, 1993).

[23] E. Cooper, A. Khare and U. Sukhatme, Phys. Rep. **251**, 267 (1995).

[24] P. Roy, B. Roy and R. Roychoudhury, Phys. Lett. **139**, 427 (1989).

[25] Q.-G. Lin, J. Phys. G **25**, 1793 (1999).